4.3 Linearly Independent Sets. Bases.

Consider the following vectors:

Example 1. Vector space $\mathbb{R}^2$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 2. Vector space $\mathbb{P}_2$

$\vec{p}_1 = 3 + x + x^2$, $\vec{p}_2 = 2 - x + 5x^2$, $\vec{p}_3 = 4 - 3x^2$.

For the set of vectors in example 1, we know how to determine if they are linearly independent or not.

For this, we consider the linear combination

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$ and equal it to $\vec{0}$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1.1)$$

Then, we determine the values of the weights or coefficients c_1, c_2, c_3, c_4. If they are all zeros then the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent.

Otherwise, the set S is linearly dependent.

We can proceed with equation (1.1) to determine $c_1, \ldots, c_4$. In fact, (1.1) leads to

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the same matrix equation that we studied in Section 4.2. Its augmented matrix can be row-reduced as

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} c_1 = 2c_3 - 4c_4 \\ c_2 = -3c_3 + 2c_4 \\ c_3, c_4 \text{ are free variables} \end{cases} \qquad (2.1)$$

It means c_3 and c_4 can take any real value. As a consequence, <u>the set S is linearly dependent.</u>

This was expected, because we know already that two vectors in $\mathbb{R}^2$ not in the same line span $\mathbb{R}^2$. In this case, we have two more vectors than needed to span $\mathbb{R}^2$.

The concepts of linear independence and linear dependence can easily be extended to general vector spaces such as the space of polynomials $\mathbb{P}_2$ of degree ≤ 2.

<u>Def.</u> – The set of vectors $S = \{\vec{v}_1, .., \vec{v}_p\}$ of a general vector space $\mathbb{V}$ are linearly independent if

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + .. + c_p \vec{v}_p = \vec{0}$$

implies $c_1 = 0, .., c_p = 0$.

Otherwise, if any of the coefficients $c_1, .., c_p$ is nonzero, the set S is linearly dependent.

So, to determine that the set of polynomials of degree ≤ 2 given in example 2 is linearly independent or not, we consider the linear combination:

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = \vec{0}$$

or

$$c_1 (3 + x + x^2) + c_2 (2 - x + 5x^2) + c_3 (4 - 3x^2) = \vec{0}$$

$$= 0 + 0x + 0x^2$$

or

$$3c_1 + 2c_2 + 4c_3 + (c_1 - c_2)x + (c_1 + 5c_2 - 3c_3)x^2$$
$$= 0 + 0x + 0x^2.$$

Which leads to the linear system of equations (*)

$$\begin{cases} 3c_1 + 2c_2 + 4c_3 = 0 \\ c_1 - c_2 \qquad\quad = 0 \\ c_1 + 5c_2 - 3c_3 = 0 \end{cases}$$

In matrix form:

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 5 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 3 & 2 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 5 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (4.1)$$

Therefore, $c_1 = 0, \ c_2 = 0, \ c_3 = 0$

And the set $S = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\} =$

$$= \{3 + x + x^2, \ 2 - x + 5x^2, \ 4 - 3x^2\}$$

is linearly independent.

(*) According to the fundamental thm of algebra a polynomial of degree 2 cannot have more than two roots unless it is the zero polynomial.

Remarks: These statements are immediate Consequences of definition.

i) If $\vec{v} \in V$ is such that $\vec{v} \neq \vec{0}$, then the set $S = \{\vec{v}\}$ is linearly independent.

ii) A set of two vectors $S = \{\vec{v}, \vec{u}\} \subset V$ is linearly dependent if and only if one of the two vectors is multiple of the other.

iii) If $\vec{0} \in S$, then S is linearly dependent.

Theorem 4. V is a vector space

A set $S = \{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\}$, with $\vec{v}_1 \neq \vec{0}$ is linearly dependent if and only if there exists $\vec{v}_j$ $(j > 1)$ such that $\vec{v}_j = c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_{j-1}\vec{v}_{j-1}$.

Proof.- $(\rightarrow)$ If S is linearly dependent, then there exists a linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_{j-1}\vec{v}_{j-1} + c_j\vec{v}_j + \overset{=0}{c_{j+1}}\vec{v}_{j+1} + \cdots + \overset{=0}{c_p}\vec{v}_p = \vec{0} \tag{5.1}$$

where not all the coefficients are zero.

Let j be the largest subscript for which $c_j \neq 0$

Obviously, $j>1$ if not $c_1\vec{v}_1=0$, which is impossible because $c_1\neq 0$ and $\vec{v}_1\neq 0$.

Therefore, Solving for $\vec{v}_j$ in equation (5.1) results

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \dots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}$$

and $\vec{v}_j$ is a linear combination of its preceding Vectors.

($\leftarrow$) If $\quad \vec{v}_j = c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}$

then $\quad c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j = \vec{0}$

As a consequence, S is linearly dependent.

The important Concept in this section is the concept of a basis for a Vector Space V.

Def.- $H\subset V$ is a Subspace of V.

The Set $B=\{\vec{b}_1,\vec{b}_2,\dots,\vec{b}_p\}$ is a basis for H if

i) B is a linearly independent set.

ii) $\operatorname{Span}(B)=H$.

Examples:

i) The set $B = \{\hat{e}_1, \hat{e}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^2$

is a basis for $\mathbb{R}^2$. They are clearly linearly indep.
and they span $\mathbb{R}^2$, because any vector
$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ can be written as $\vec{x} = x\hat{e}_1 + y\hat{e}_2$.
So any $\vec{x} \in \mathbb{R}^2$ is a linear combination of $\hat{e}_1$ and $\hat{e}_2$.

ii) Similarly,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \overset{n}{\cdots} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^n.$$

This basis is called the standard basis for $\mathbb{R}^n$.

iii) Consider $\mathbb{P}_n$ (polynomials of degree $\leq n$).

$B = \{1, x, x^2, \dots, x^n\} \subset \mathbb{P}_n$ is a basis for $\mathbb{P}_n$.
(STANDARD BASIS)

It spans $\mathbb{P}_n$, because any polynomial in $\mathbb{P}_n$
can be written as

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

(Fundam. Thm Algebra)

It's also linearly indep. because the only possibility for
$c_0 \cdot 1 + c_1 x + \cdots + c_n x^n = 0(x) = 0 \cdot 1 + 0 \cdot x + \cdots + 0 \cdot x^n$ is $c_0 = 0, c_1 = 0, \dots, c_n = 0$.

Remarks:

i) Any set of n linearly independent vectors in $\mathbb{R}^n$

$$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset \mathbb{R}^n \text{ is a basis for } \mathbb{R}^n.$$

We only need to prove that B spans $\mathbb{R}^n$.

But, this is trivially true from the invertible matrix theorem because if we define the matrix A as

$$A = [\vec{v}_1 \; \vec{v}_2 \ldots \vec{v}_n]$$

then any $\vec{b} \in \mathbb{R}^n$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ because the linear system

$$A\vec{x} = \vec{b}, \quad \text{always has a solution.}$$

ii) The set of polynomials

$$B = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \{3+x+x^2, 2-x+5x^2, 4-3x^2\} \subset \mathbb{P}_2$$

is a basis for $\mathbb{P}_2$.

We already proved that B is linearly independent.

Prove that $\quad \operatorname{Span}(B) = \mathbb{P}_2$.

Show

Span $(B) = \mathbb{P}_2$

Consider an arbitrary polynomial $\vec{q} \in \mathbb{P}_2$

$$\vec{q} = d_0 + d_1 x + d_2 x^2, \qquad d_0, d_1, d_2 \in \mathbb{R}.$$

Want to prove $\vec{q} \in$ Span (B)

Equivalent to find C_1, C_2 and C_3 such that

$$C_1 \overbrace{(3 + x + x^2)}^{\vec{p}_1} + C_2 \overbrace{(2 - x + 5x^2)}^{\vec{p}_2} + C_3 \overbrace{(4 - 3x^2)}^{\vec{p}_3}$$

$$= d_0 + d_1 x + d_2 x^2$$

Which leads to

$$\begin{cases} 3C_1 + 2C_2 + 4C_3 = d_0 \\ C_1 - C_2 = d_1 \\ C_1 + 5C_2 - 3C_3 = d_2 \end{cases}$$

or in matrix form:

$$\underbrace{\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 5 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}}_{\vec{C}} = \underbrace{\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}}_{\vec{d}}$$

Since $A \sim I_3$ (page 4 Equ. (4.1))

A is invertible and the matrix equation $A\vec{C} = \vec{d}$ has a solution. Therefore, $\vec{q} \in$ Span (B). Moreover, this solution is unique.

Spanning Set theorem

Thm 5. $S = \{\vec{v}_1, .., \vec{v}_p\} \subset V$ (Vector space).

$H = \text{Span}\{\vec{v}_1, .., \vec{v}_p\}$. then

a) If $\vec{v}_k \in S$ is a linear-combination of the others, i.e.,

$$\vec{v}_k = c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots + c_p\vec{v}_p$$

then $\text{Span}\{\vec{v}_1, .., \vec{v}_{k-1}, \vec{v}_{k+1}, .., \vec{v}_p\} = H$.

b) If $H \neq \{\vec{0}\}$ then there is $B \subset H$ such that B is a basis for H.

Proof:- a) We want to prove that any $\vec{x} \in H$ can be written as

$$\vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_{k-1}\vec{v}_{k-1} + d_{k+1}\vec{v}_{k+1} + \cdots + d_p\vec{v}_p \tag{9.1}$$

In fact, $\vec{x} \in H$ if and only if

$$\vec{x} = a_1\vec{v}_1 + \cdots + a_{k-1}\vec{v}_{k-1} + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} + \cdots + a_p\vec{v}_p$$

Or
$$\vec{x} = a_1\vec{v}_1 + \cdots + a_{k-1}\vec{v}_{k-1} + \overbrace{a_k(c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots c_p\vec{v}_p)}^{a_k\vec{v}_j}$$

$$+ a_{k+1}\vec{v}_{k+1} + \cdots + a_p\vec{v}_p =$$

$$= (a_1 + a_k c_1)\vec{v}_1 + \cdots + (a_{k-1} + a_k c_{k-1})\vec{v}_{k-1} + (a_{k+1} + a_k c_{k+1})\vec{v}_{k+1} + \cdots$$

$$+ \cdots + (a_p + a_k c_p)\vec{v}_p$$

by renaming $a_i + a_k c_i = d_i$, $i = 1, .., k-1, k+1, .., p$.

We obtain (9.1), and $\vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2, .., \vec{v}_{k-1}, \vec{v}_{k+1}, .., \vec{v}_p\}$

b) If the spanning set of H, $S = \{\vec{v}_1, .., \vec{v}_p\}$ is linearly independent then S is a basis for H. otherwise, we use the procedure outlined in part (a) of this theorem. and remove vectors successively until the new set becomes linearly independent. This is possible because $H \neq \{\vec{0}\}$ implies that at least one of $\vec{v}_j \neq \vec{0}$, $j = 1, .., p$.

Apply this procedure to our Example 1 and show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $Col(A)$ where $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$.

Finding a basis for Null(A).

In Section 4.2, we found the null space for the matrix

$$B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$$

To do this, we considered the homogeneous linear system $B\vec{x} = \vec{0}$

row-reduced the augmented matrix and found out that any vector $\vec{x} \in Null(B)$ is a linear combination of the following vectors in $\mathbb{R}^4$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

It means

$$Null(B) = Span\left\{ \overbrace{\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}}^{=\vec{v}_1}, \overbrace{\begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}}^{=\vec{v}_2} \right\}$$

these two vectors form a linearly independent set, because they are not multiples of each other. Therefore, the set

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ form a basis for $Null(B)$.

This procedure followed to find Null(B) always leads to a basis for Null(B) for any matrix B.

Finding a basis for Col (A)

Consider again

$$\begin{array}{cc} & \begin{array}{cccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \end{array} \\ B = & \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -2 \end{bmatrix} = \tilde{B}$$

$$= [\vec{d}_1 \ \vec{d}_2 \ \vec{d}_3 \ \vec{d}_4]$$

Clearly,

$$\vec{d}_3 = -2\vec{d}_1 + 3\vec{d}_2$$

$$\vec{d}_4 = 4\vec{d}_1 - 2\vec{d}_2$$

Moreover, the set $S = \{\vec{d}_1, \vec{d}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly indep.

An important observation here is that the same linear dependence relationship among the column vectors of $\tilde{B}$ occurs among the column vectors of B. It means

$$\vec{b}_3 = -2\vec{b}_1 + 3\vec{b}_2, \quad \vec{b}_4 = 4\vec{b}_1 - 2\vec{b}_2$$

and also the set $R = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

Why is this true in general?

Therefore,

$$Col(B) = Span\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}.$$

$$\overset{\text{Spanning set thm.}}{=\!=\!=} Span\{\vec{b}_1, \vec{b}_2\}$$

and $R = \{\vec{b}_1, \vec{b}_2\}$ is linearly indep.

Therefore, R is a basis for $Col(B)$.

The procedure applied in the previous example can be applied to any matrix A to find a basis for $Col(A)$. This procedure is supported by the following thm.

<u>Thm 6</u> The pivot columns of a matrix A form a basis for $Col(A)$.

Before proving thm 6, we will prove some lemmas

Important observations about Nul(B) and Col(B)

$$B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}, \qquad B_{2\times 4}$$

$$\vec{x} \in Nul(B) \iff B\vec{x} = \vec{0}$$

$$\vec{x} \in Col(B) \iff \vec{x} \in Span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

Computation of Nul(B)

Based on the statement $B \sim \tilde{B}$ then $B\vec{x} = \vec{0} \iff \tilde{B}\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 2x_3 - 4x_4 \\ x_2 = -3x_3 + 2x_4 \\ x_3, x_4 \text{ free} \end{array}$$

Therefore,

$$\vec{x} \in Nul(B) \iff \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

x_3, x_4 arbitrary real numbers

So $$Nul(B) = Span\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^4$$

If we consider the linear transformation

$$B_{2\times 4} \qquad T: \mathbb{R}^4 \longrightarrow \mathbb{R}^2, \quad \vec{x} \longrightarrow B\vec{x}$$

We clearly see

Domain $T = \mathbb{R}^4$, $Nul(B) \subset \text{Dom } T$

Range $T \subseteq \mathbb{R}^2$, $Col(B) = \text{Range } T$.

Relationship between Nul (A) and Nul($\tilde{A}$).

Lemma 1.- Elementary row operations do not change the null space of a matrix A.

Equivalent to say:

If $A \sim \tilde{A}$ then $\boxed{Nul(A) = Nul(\tilde{A})}$

or $A\vec{x} = \vec{0} \Leftrightarrow \tilde{A}\vec{x} = \vec{0}$

Proof.- If $A \sim \tilde{A}$, then $\tilde{A} = E_r \dots E_1 A$,

Where E_i $(i=1,..,r)$ are elementary matrices.

Thus, if $\vec{x} \in Nul(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow$

$$\tilde{A}\vec{x} = (E_r \dots E_1 A)\vec{x} = (E_r E_{r-1} \dots E_1)\underbrace{(A\vec{x})}_{=\vec{0}} = \vec{0}$$

$\Rightarrow x \in Nul(\tilde{A}) \Rightarrow Nul(A) \subset Nul(\tilde{A})$

Similarly,

if $\vec{x} \in Nul(\tilde{A}) \Rightarrow \tilde{A}\vec{x} = \vec{0} \Rightarrow$

$$A\vec{x} = (E_1^{-1} \dots E_r^{-1}\tilde{A})\vec{x} = (E_1^{-1} \dots E_r^{-1})\underbrace{\tilde{A}\vec{x}}_{=\vec{0}} = \vec{0}.$$

$\Rightarrow \vec{x} \in Nul(A). \Rightarrow Nul(\tilde{A}) \subset Nul(A).$

Remark: The practical value of this thm is that to find Null(A) for a given matrix A, first row-reduce A to $\tilde{A}$ and then find Null($\tilde{A}$) which is the same as Nul(A).

PROCEDURE TO FIND A BASIS FOR COL(A).

Consider the matrix $A_{4\times5}$

$$A = \begin{array}{c} \begin{matrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix} \end{array} \sim \begin{array}{c} \begin{matrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \tilde{A}$$

Obviously, $\boxed{Col(A) \neq Col(\tilde{A})}$

For instance, $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \notin Col(\tilde{A})$ why not?

However, $\vec{b}_2 = 2\vec{b}_1$, $\vec{a}_2 = 2\vec{a}_1$

$\vec{b}_4 = \vec{b}_2 - 2\vec{b}_3$, $\vec{a}_4 = \vec{a}_2 - 2\vec{a}_3$

Also, the set $\tilde{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{b}_3}{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{b}_5}{\begin{bmatrix} 5 \\ 3 \\ -7 \\ 0 \end{bmatrix}} \right\}$ is lin. indep.

because none of its vectors is lin. comb. of the preceding ones.

Therefore, $Span(\tilde{B}) = Col(\tilde{A})$, and $\tilde{B}$ is a basis for $Col(\tilde{A})$.

It can be shown that the set

$B = \left\{ \underset{\vec{a}_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\vec{a}_3}{\begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}}, \underset{\vec{a}_5}{\begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix}} \right\}$ is also lin indep.

and $Span(B) = Col(A)$. It means B is a basis for $Col(A)$.

$$A = [\vec{a}_1\, \vec{a}_2 \ldots \vec{a}_n], \quad \tilde{A} = [\vec{b}_1\, \vec{b}_2 \ldots \vec{b}_n]$$

Lemma 2. If $A \sim \tilde{A}$ then

i) Columns of A have exactly the same linear dependence relationship as the columns of $\tilde{A}$.
For instance,
a) A subset of column vectors of A is linearly independent if the corresponding column vectors of $\tilde{A}$ form a set linearly independent.

ii) A subset of column vectors of A forms a basis for $Col(A)$ if and only if the corresponding column vectors of $\tilde{A}$ form a basis for $Col(\tilde{A})$.

Proof.– From lemma 1, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution of

$A\vec{x} = \vec{0}$ or $\boxed{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = 0}$ (16.1)

if and only if $\vec{x}$ is a solution of

$\tilde{A}\vec{x} = \vec{0}$ or $\boxed{x_1 \vec{b}_1 + x_2 \vec{b}_2 + \cdots + x_n \vec{b}_n = \vec{0}}$ (16.2)

Therefore, $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is linearly independent if and only if $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is also linearly independent

Also, $\vec{b}_j$ $(j=1,.,n)$ is linearly dependent on $\vec{b}_1,.,\vec{b}_{j-1},\vec{b}_{j+1},..\vec{b}_n$ if and only if $\vec{a}_j$ is linearly dependent on $\vec{a}_1,..,\vec{a}_{j-1},\vec{a}_{j+1},...,\vec{a}_n$

Moreover, the linear dependence relationship is identical in both cases, because the coefficients (or components of $\vec{x}$) are exactly the same in (16.1) and (16.2).

ii) If a subset B of the columns of A forms a basis for column A, then B is linearly independent and the corresponding subset $\tilde{B}$ of the columns of $\tilde{A}$ is also linearly independent from part (i).

Also, $\operatorname{Span}(B) = \operatorname{Col}(A)$, then any linear combination including another vector of the columns of A will be linearly dependent.

From (i), this same property is true for the columns of $\tilde{A}$. As a consequence

$$\operatorname{Span}(\tilde{B}) = \operatorname{Col}(\tilde{A}),$$

and $\tilde{B}$ is a basis for $\tilde{A}$.

(The reciprocal is also true
If $\tilde{B}$ is a basis for $\operatorname{Col}(\tilde{A})$
$\Rightarrow$ B is a basis for $\operatorname{Col}(A)$.
identical proof.)

Proof. of theorem 6

"Pivot columns of a matrix A form a basis for Col(A)."

First, we consider the row-reduced matrix $\tilde{A}$

$$\tilde{A} = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n]$$

If $\vec{b}_j$ is the first pivot column, then the next pivot column $\vec{b}_p$ is linearly indep. with $\vec{b}_j$ because it has its leading "1" in the row below to where $\vec{b}_j$ has its leading "1".

$$\begin{array}{c} \begin{array}{ccc} \vec{b}_j & & \vec{b}_p \end{array} \\ \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ 1 & & 0 \\ 0 & & 1 \\ \vdots & & 0 \\ 0 & & \vdots \\ & & 0 \end{array}\right] \end{array}$$

The next pivot column will also be linearly indep. with the preceding two columns. This process can be continued until there are not more pivot columns.

Then, the set obtained $\tilde{B} = \{\vec{b}_j, \vec{b}_p, \dots, \vec{b}_q\}$ is linearly independent.

Moreover, any column in between is linearly dependent on the previous ones.

Therefore,

$$\text{Span}(\tilde{B}) = \text{Col}(\tilde{A}).$$

And $\tilde{B}$ is a basis for $\text{Col}(\tilde{A})$.

From the previous lemma 2, the corresponding set B of columns of A is also a basis for $\text{Col}(A)$. This set B is by definition of pivot column formed by the pivot columns of matrix A, which is what we want to prove.